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# On the Core of Routing Games with Revenues\*

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## Abstract

Traveling salesman problems with revenues form a generalization of traveling salesman problems. Here, next to travel costs an explicit revenue is generated by visiting a city. We analyze routing problems with revenues, where a predetermined route on all cities determines the tours along subgroups. Corresponding routing games with revenues are analyzed. It is shown that these games have a nonempty core and a complete description of the core is provided.

**Keywords:** Routing problems, revenues, core.

**JEL Classification Numbers:** C71

## 1 Introduction

In a traveling salesman (TS) situation a salesman, starting in his home city, has to visit a set of cities exactly once and has to come back to its home city at the end of the journey. Associating travel costs to connections the problem is how to find a tour with minimal cost. It is known that TS problems are NP-hard in general. For a survey on TS problems we refer to Lawler, Lenstra, Kan, Shmoys and Hurkens (1997).

Fishburn and Pollak (1983) introduced the cost allocation problem that arises when each city (except the home city) corresponds to a player. The cost allocation is concerned with a fair allocation of the joint costs of the cheapest tour. This cost allocation problem was first studied within the framework of game theory by Potters, Curiel and Tijs (1992) by introducing TS games. In a TS game, the value of a coalition of cities is the

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value of the cheapest tour in the TS problem associated to the coalition. Here, only (and exactly) the cities in the coalition will be visited. If the triangular inequalities are satisfied every 3-, 4-, and 5-person TS game has a nonempty core (cf. Potters et al. (1992), Tamir (1989), and Kuipers (1993)). In the same setting however Tamir (1989) provides an example of a 6-person TS game with an empty core. In Estévez-Fernández, Borm and Hamers (2003) it is seen that these results can be generalized to multiple (longest) traveling salesman (M(L)TS) games. In an MTS problem the salesman has to visit each city exactly once except for the home city which can be revisited as many times as desired. In a longest traveling salesman (LTS) problem there are profits associated to connections instead of travel costs. Hence, the objective of an LTS problem is to find a tour with maximal profit.

In Potters et al. (1992) also the class of fixed routing games is introduced. Here, the route along all cities is predetermined (e.g. by restrictions in the agenda of the salesman) and this tour determines the tours along all possible coalitions. The value of a coalition of a routing game is defined as the cost associated to the tour that visits the members of the coalition in the same relative order as in the predetermined tour. Potters et al. (1992) show that routing games have a nonempty core if the predetermined tour is an optimal tour for the related TS problem. Derks and Kuipers (1997) give a time efficient algorithm to provide core elements of a routing game.

This paper studies routing problems with revenues (RR-problems): next to travel costs and a predetermined route on all cities, revenues of a visit are explicitly modeled and taken into account. Note that since the revenues obtained by the visit of a salesman are explicitly given, it might be the case that some of the cities will not be visited by the salesman if the objective is to maximize total joint profit. We will assume that the predetermined route is optimal in this sense and indeed visits all cities. Still, it might be optimal for a coalition not to visit all its cities in the prescribed relative order. Hence, the value of the associated routing game for a specific coalition is defined as the maximum attainable profit by one of its subcoalitions if the salesman visits all cities in this subcoalition in the relative order given by the predetermined route on all cities. We will show that every routing game has a nonempty core. Moreover, a complete description of the core is provided.

The idea of analyzing cost problems arising from a general service facility by taking explicitly into account the profits that the service will generate is not new. It was first studied in Littlechild and Owen (1976) within the framework of airport problems, with a more recent follow up in Brânzei, Iñarra, Tijs and Zarzuelo (2003). Meertens and Potters (2004) consider fixed tree games with revenues. Suijs, Borm, Hamers, Quant and Koster (2005) study the sharing of costs and revenues within a public network communication structure.

## 2 Routing games with revenues

In this section we introduce routing games with revenues.

Let  $N = \{1, 2, \dots, n\}$  denote the set of cities that a traveling salesman has to visit. Let  $N_0 := N \cup \{0\}$  where 0 denotes the traveling salesman's home city. Let  $C = (c_{ij})$  be an  $N_0 \times N_0$ -matrix where  $c_{ij}$  represents the costs to go from city  $i$  to city  $j$ . Throughout this article we will assume that:

- (i)  $c_{ii} = 0$  for all  $i \in N_0$ ,
- (ii)  $c_{ij} = c_{ji}$  for all  $i, j \in N_0$ , (symmetry)
- (iii)  $c_{ij} \leq c_{ik} + c_{kj}$  for all  $i, j, k \in N_0$ . (triangle inequalities)

Whenever city  $i \in N$  is visited by the traveling salesman, a revenue  $b_i \geq 0$  is obtained. Due to the explicit modeling of the revenues we assume that the salesman, who starts from city 0, will visit each city *at most once*, and only returns to city 0 at the end of the journey.

Let  $R \subset N$  and set  $R_0 := R \cup \{0\}$ . A bijection  $\pi : R_0 \rightarrow R_0$  is called a *cyclic permutation* if  $\min\{t \in \mathbb{N} \mid \pi^t(i) = i\} = |R| + 1$  for every  $i \in R_0$ . We will denote by  $\Pi(R)$  the set of all cyclic permutations on  $R_0$ . A cyclic permutation  $\pi$  corresponds to a *tour along  $R$* : it starts in 0 and visits each city in  $R$  exactly once returning to 0 at the end of the trip. Here,  $\pi(i)$  is the city immediately visited after city  $i$  for all  $i \in R_0$ . For convenience, we will sometimes denote city 0 also by  $n + 1$  and in particular  $\pi(i) = n + 1$  means that  $i$  is the last city on the tour. For  $\pi \in \Pi(R)$ , we denote by  $c(\pi, R)$  the cost associated to the tour induced by  $\pi$ , i.e.,  $c(\pi, R) = \sum_{i \in R_0} c_{i\pi(i)}$ . Consequently, the minimal cost  $c(R)$  of a tour along  $R$  is given by

$$c(R) := \min_{\pi \in \Pi(R)} \{c(\pi, R)\}.$$

The total profit  $p(R)$  obtained when the salesman has visited all cities in  $R$  according to a tour with minimal cost is

$$p(R) := b(R) - c(R),$$

where  $b(R) := \sum_{i \in R} b_i$ .

Due to the revenue structure it may be more profitable for  $N$  not to make a (complete) tour on  $N$  itself but on a subset  $R \subset N$ , leaving  $N \setminus R$  unvisited. Therefore, the optimization problem for  $N$  boils down to finding a subset of cities  $R$  such that  $p(R)$  is maximal. We denote the maximal profit for  $N$  by  $v(N)$ , i.e.,

$$v(N) := \max_{R \subset N} \{p(R)\}.$$

From now on we will assume that it is optimal to visit all cities in  $N$  via the cyclic permutation  $\hat{\pi} \in \Pi(N)$ . We will also assume without loss of generality that

$$\text{the optimal order } \hat{\pi} \text{ for } N \text{ is given by } 0 - 1 - 2 - \dots - n - 0. \quad (2.1)$$

Hence,  $v(N) = p(N)$ .

Associating each city in  $N$  with a player, the question we would like to address is how to share  $v(N)$  among the players. For this we choose the “routing” approach, where  $\hat{\pi}$  determines the order in which potential subcoalitions are visited.

For  $S \subset N$ , the cyclic permutation  $\hat{\pi}_S \in \Pi(S)$  induced by  $\hat{\pi}$  is obtained from  $\hat{\pi}$  by skipping the cities in  $N \setminus S$  and leaving the order of the remaining cities unchanged. Formally,  $\hat{\pi}_S$  is given by

$$\hat{\pi}_S(i) = \hat{\pi}^{t(i)}(i) \text{ for every } i \in S_0$$

where  $t(i) := \min\{t \in \mathbb{N} \mid \hat{\pi}^t(i) \in S_0\}$ . With a minor abuse of notation we will denote  $c(\hat{\pi}_S, S)$  by  $c(\hat{\pi}, S)$ .

A coalition  $S \subset N$  need not decide on the complete tour  $\hat{\pi}_S$  on  $S$ : a tour  $\hat{\pi}_R$  on a subset  $R \subset S$  may be more profitable. Hence, we define the value  $v_{\hat{\pi}}(S)$  in the *routing game*  $(N, v_{\hat{\pi}})$  by

$$v_{\hat{\pi}}(S) = \max_{R \subset S} \{b(R) - c(\hat{\pi}, R)\}$$

Note that  $v_{\hat{\pi}}(N) = v(N)$ .

**Example 2.1.** Consider the routing problem with revenues represented in Figure 1 where the numbers at the edges represent the travel costs and the boldface numbers at the nodes represent the revenues.

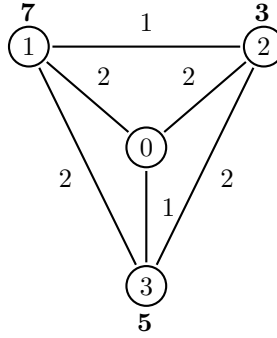


Figure 1: The routing problem with revenues in Example 2.1.

Note that assumption (2.1) is satisfied. The associated routing game with revenues has values:  $v_{\hat{\pi}}(\{1\}) = 3$ ,  $v_{\hat{\pi}}(\{2\}) = 0$ ,  $v_{\hat{\pi}}(\{3\}) = 3$ ,  $v_{\hat{\pi}}(\{1, 2\}) = 5$ ,  $v_{\hat{\pi}}(\{1, 3\}) = 7$ ,  $v_{\hat{\pi}}(\{2, 3\}) = 3$  and  $v_{\hat{\pi}}(N) = 9$ .  $\diamond$

Note that if the revenues are so high that the salesman will visit all cities of every coalition, then the routing game with revenues is strategically equivalent<sup>1</sup> to a routing game à la Potters et al. (1992) and Derks and Kuipers (1997).

### 3 The Core

In this section we will study the core of routing games with revenues. We will show that routing games with revenues have a nonempty core. Moreover, we will give an intuitive interpretation of all core elements.

Let  $(N, v)$  be a cooperative game. Recall that the *core* of  $(N, v)$  is given by

$$\text{Core}(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N), x(S) \geq v(S) \text{ for all } S \in 2^N\},$$

i.e., the core is the set of efficient allocations of  $v(N)$  such that there is no coalition with an incentive to split off.

In the following example we illustrate that taking into account revenues has a definite impact on the structure of the core.

**Example 3.1.** Consider the routing problem with revenues represented in Figure 2.

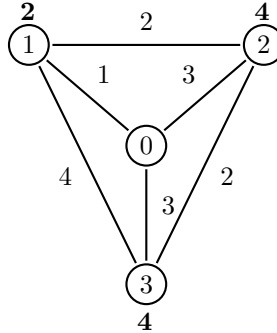


Figure 2: The routing problem with revenues in Example 3.1.

It is readily checked that assumption (2.1) is satisfied and that  $v_{\hat{\pi}}(S) = 0$  for every  $S \subset N$ ,  $S \neq N$ , and  $v_{\hat{\pi}}(N) = 2$ . Therefore,

$$\text{Core}(v_{\hat{\pi}}) = \text{conv}\{(2, 0, 0), (0, 2, 0), (0, 0, 2)\}.$$

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<sup>1</sup>Here, we make a slight abuse of language when we say that a routing game with revenues is strategically equivalent to a (cost) routing game. We mean that there exist  $k \in \mathbb{R}_{++}$ ,  $a \in \mathbb{R}^N$  and  $(N, c)$ , a (cost) routing game, such that  $v_{\hat{\pi}}(S) = a(S) - kc(S)$  for every  $S \subset N$ .

Consider now the associated (cost) routing game à la Potters et al. (1992) and Derks and Kuipers (1997) in which the revenues are explicitly not considered: they are high enough. One readily verifies that  $c_{\hat{\pi}}(\{1\}) = 2$ ,  $c_{\hat{\pi}}(\{2\}) = 6$ ,  $c_{\hat{\pi}}(\{3\}) = 6$ ,  $c_{\hat{\pi}}(\{1, 2\}) = 6$ ,  $c_{\hat{\pi}}(\{1, 3\}) = 8$ ,  $c_{\hat{\pi}}(\{2, 3\}) = 8$  and  $c(N) = 8$ . Here,

$$\text{Core}(c_{\hat{\pi}})^2 = \text{conv}\{(2, 0, 6), (0, 2, 6), (0, 6, 2), (2, 4, 2)\}.$$

Hence, there is not an obvious relation between  $\text{Core}(c_{\hat{\pi}})$  and  $\text{Core}(v_{\hat{\pi}})$ . Moreover, it is readily checked that the above routing game with revenues is not strategically equivalent to any routing game à la Potters et al. (1992) and Derks and Kuipers (1997).  $\diamond$

Next, we will show that a routing game with revenues,  $(N, v_{\hat{\pi}})$ , corresponding to travel cost matrix  $C \in \mathbb{R}^{N_0 \times N_0}$  and revenue vector  $b \in \mathbb{R}^N$  has a nonempty core.

For every  $S \subset N$  we define the linear programming problem  $\text{LP}(S)$  by:

$$\text{maximize} \quad \sum_{i=0}^n \sum_{j=i+1}^{n+1} (b_j - c_{ij})x_{ij} \tag{3.1}$$

$$\text{s.t.} \quad \sum_{k=0}^{i-1} x_{ki} \leq e_i^S \quad \text{for all } i \in \{1, \dots, n\}; \tag{3.2}$$

$$\sum_{k=0}^{i-1} x_{ki} - \sum_{j=i+1}^{n+1} x_{ij} = 0 \quad \text{for all } i \in \{1, \dots, n\}; \tag{3.3}$$

$$x_{ij} \geq 0 \quad \text{for all } i, j \in \{0, 1, \dots, n, n+1\} \text{ with } i < j. \tag{3.4}$$

with  $b_0 = b_{n+1} := 0$  and  $c_{in+1} := c_{i0}$  and where  $e^S \in \mathbb{R}^N$  is a vector of zeros and ones with  $e_i^S = 1$  if  $i \in S$  and  $e_i^S = 0$  otherwise.

It is readily checked that  $\text{LP}(S)$  is feasible and bounded. Here,  $x_{ij}$  can be interpreted as the “amount of flow that goes from  $i$  to  $j$ ”. The profit obtained per unit of flow from  $i$  to  $j$  is  $b_j - c_{ij}$  for every  $i$  and  $j$  such that  $1 \leq i < j \leq n+1$  and the objective function is to maximize the total profit as represented in (3.1). Equation (3.2) indicates that the total flow “arriving” at city  $i$  can not exceed one unit of flow, i.e., one can think of this as a capacity restriction on the nodes. Equation (3.3) makes sure that the amount of flow “arriving” at  $i$  equals the amount of flow “leaving”  $i$ .

Note that the game  $(N, u)$  with  $u(S)$  defined as the optimal value of  $\text{LP}(S)$  for  $S \subset N$ , is a linear production game and therefore it has a nonempty core (cf. Owen (1975)).

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<sup>2</sup>The core of a cost game is defined as  $\text{Core}(c) = \{x \in \mathbb{R}^N \mid x(N) = v(N), x(S) \leq c(S) \text{ for all } S \in 2^N\}$ .



**Lemma 3.1.**

- (i)  $u(S) \geq v_{\hat{\pi}}(S)$  for every  $S \subset N$ .
- (ii)  $u(N) = v_{\hat{\pi}}(N)$ .

PROOF: (i) Let  $S \subset N$  and  $R \subset S$  such that

$$v_{\hat{\pi}}(S) = b(R) - c(\hat{\pi}, R).$$

If  $R = \emptyset$ , then  $v_{\hat{\pi}}(S) = 0 \leq u(S)$ . Otherwise,  $R := \{i_1, \dots, i_r\}$  with  $i_1 < \dots < i_r$ . Define

$$x_{ij} := \begin{cases} 1 & \text{if } i \in \{0, i_1, \dots, i_{r-1}\} \text{ and } j = \hat{\pi}(i), \text{ or } i = i_r \text{ and } j = n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then,  $(x_{ij})_{0 \leq i < j \leq n+1}$  is a feasible solution of  $\text{LP}(S)$  and

$$u(S) \geq \sum_{i=0}^n \sum_{j=i+1}^{n+1} (b_j - c_{ij})x_{ij} = b_{i_1} + \dots + b_{i_r} - (c_{0i_1} + c_{i_1i_2} + \dots + c_{i_{r-1}i_r} + c_{0i_r}) = v_{\hat{\pi}}(S).$$

(ii) By (i) it suffices to show that  $u(N) \leq v_{\hat{\pi}}(N)$ . Note that  $\text{LP}(N)$  is a transportation problem with  $\{0, 1, \dots, n\}$  the set of sources,  $\{1, \dots, n, n+1\}$  the set of sinks, and such that there are no links going from a source  $i$  to a sink  $j$  with  $i > j$  and the reward when going from  $i \in N$  to itself is zero. Then, there exists an integral optimal solution,  $\bar{x} = (\bar{x}_{ij})_{0 \leq i < j \leq n+1}$ , for  $\text{LP}(N)$  (see Nemhauser and Wolsey (1988), Chapter I.3, Corollary 5.2). Moreover,

- (i)  $\bar{x}_{ij} \in \{0, 1\}$  for every  $i, j$  with  $0 \leq i < j \leq n+1$  by equations (3.2) and (3.4).
- (ii) If  $\sum_{k=0}^{i-1} \bar{x}_{ki} = 1$ , then there exists a unique  $k(i) \in \{0, \dots, i-1\}$  such that  $\bar{x}_{k(i)i} = 1$  by (i).
- (iii) If  $\sum_{k=0}^{i-1} \bar{x}_{ki} = 1$ , then there exists a unique  $j(i) \in \{i+1, \dots, n+1\}$  such that  $\bar{x}_{ij(i)} = 1$  by equation (3.3).
- (iv)  $\sum_{i=1}^n \bar{x}_{0i} = \sum_{i=1}^n \bar{x}_{in+1}$  by equation (3.3).

Let  $N(\bar{x}) := \{i \in N \mid \sum_{k=0}^{i-1} \bar{x}_{ki} = 1\}$  and let  $\{i_1, \dots, i_r\} = \{i \in N(\bar{x}) \mid \bar{x}_{0i} = 1\}$ . Let  $t(i_l)$  be the smallest integer such that  $j^{t(i_l)}(i_l) = n+1$ . We define

$$N_l(\bar{x}) = \{j^{t(i_l)}(i_l) \mid t \in \{1, \dots, t(i_l) - 1\}\}$$

for  $l \in \{1, \dots, r\}$ . It is readily checked that  $N_1(\bar{x}), \dots, N_r(\bar{x})$  is a partition of  $N(\bar{x})$ . Moreover, note that this partition implies that  $r$  tours,  $\pi^1 \in \Pi(N_1(\bar{x})), \dots, \pi^r \in \Pi(N_r(\bar{x}))$ , will be followed in the optimal solution, where tour  $\pi^l$  is given by  $0 - i_l - \dots - j^{t(i_l)-1}(i_l) - (n+1)$  for  $l \in \{1, \dots, r\}$ .

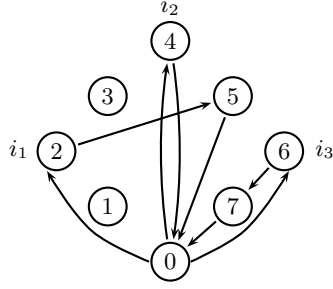


Figure 3: Representation of the tours given by  $\pi^l$ .

Define  $\tilde{\pi} \in \Pi(N(\bar{x}))$  by  $0 - i_1 - \dots - j^{t(i_1)-1}(i_1) - i_2 - \dots - j^{t(i_2)-1}(i_2) - \dots - i_r - \dots - j^{t(i_r)-1}(i_r) - (n+1)$ .

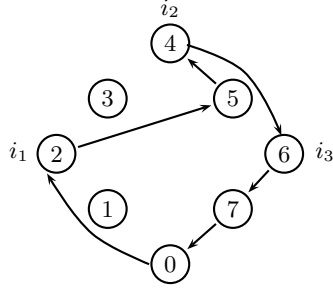


Figure 4: Representation of  $\tilde{\pi}$ .

Hence,

$$\begin{aligned}
u(N) &= \sum_{i=0}^{n+1} \sum_{j=i+1}^{n+1} (b_j - c_{ij}) \bar{x}_{ij} \\
&= \sum_{l=1}^r \sum_{i \in N_l(\bar{x}) \cup \{0\}} (b_{\pi^l(i)} - c_{i\pi^l(i)}) \\
&= \sum_{l=1}^r b(N_l(\bar{x})) - \sum_{l=1}^r c(\pi^l, N_l(\bar{x})) \\
&\leq b(\cup_{l=1}^r N_l(\bar{x})) - c(\tilde{\pi}, \cup_{l=1}^r N_l(\bar{x})) \\
&\leq b(\cup_{l=1}^r N_l(\bar{x})) - c(\cup_{l=1}^r N_l(\bar{x})) \\
&\leq b(N) - c(N) = v_{\tilde{\pi}}(N)
\end{aligned}$$

where the first inequality holds by the triangular inequalities, the second one holds by definition of  $c(\cup_{l=1}^r N_l(\bar{x}))$  and the last one by assumption (2.1).  $\square$

Note that if two games  $(N, v)$  and  $(N, u)$  are such that  $v(S) \leq u(S)$  for every  $S \subset N$ ,  $v(N) = u(N)$ , and  $\text{Core}(u) \neq \emptyset$ , then  $\text{Core}(v) \neq \emptyset$  and  $\text{Core}(u) \subset \text{Core}(v)$ . Hence, as a direct consequence of Lemma 3.1 we have

that a routing game with revenues has a nonempty core.

**Theorem 3.2.** Any routing game with revenues has a nonempty core.

The following result gives a full description of the core of a routing game with revenues. It states that an allocation  $x$  belongs to the core of the game if each coordinate  $x_i$  can be written as  $x_i = b_i - c_{i-1i} + z_{i-1} - z_i$ . This can be interpreted in the following way: First of all, player  $i$  obtains the revenue  $b_i$  when the salesman visits its city and has to pay the travel costs  $c_{i-1i}$  from city  $i-1$  to city  $i$ . Next, since player  $i-1$  also gets revenues from the visit, it will help player  $i$  with the travel costs by paying a compensation  $z_{i-1}$ . In a similar way, player  $i$  will help player  $i+1$  with the travel costs of the trip from city  $i$  to city  $i+1$  with  $z_i$ . Equation (3.6) below reflects that player  $i$  will never compensate  $i+1$  more than the total amount he gets once  $i-1$  has paid the compensation. Equation (3.7) reflects the fact that player  $j+1$  indeed prefers that the salesman comes from player  $j$  instead of another player  $i (< j)$ .

**Theorem 3.3.** Let  $(N, v_{\hat{\pi}})$  be a routing game with revenues corresponding to travel cost matrix  $C \in \mathbb{R}^{N_0 \times N_0}$  and revenue vector  $b \in \mathbb{R}^N$ . Then, the following three assertions are equivalent.

- (i)  $x \in \text{Core}(v_{\hat{\pi}})$ .
- (ii)  $x \geq 0$ ,  $x(N) = v_{\hat{\pi}}(N)$ , and  $x(N \setminus S) \geq v_{\hat{\pi}}(N \setminus S)$  for every  $S \subset N$  with  $S = \{i, i+1, \dots, j\}$  and  $i \leq j$ .
- (iii)  $x_i := b_i - c_{i-1i} + z_{i-1} - z_i$  for all  $i \in \{1, \dots, n\}$  with

$$z_0 := 0, \quad z_n := c_{0n} \tag{3.5}$$

$$z_i - z_{i-1} \leq b_i - c_{i-1i} \quad \text{for all } i \in \{1, \dots, n\} \tag{3.6}$$

$$z_j - z_i \geq c_{jj+1} - c_{ij+1} \quad \text{for all } i, j \in \{0, 1, \dots, n\} \text{ with } i < j. \tag{3.7}$$

PROOF: (i) $\Rightarrow$ (ii) is immediate and therefore omitted.

(ii) $\Rightarrow$ (iii) Let  $x \in \mathbb{R}^N$  satisfy the conditions mentioned in assertion (ii) of the theorem. Define the vector  $z \in \mathbb{R}^{N_0}$  as follows:

$$\begin{aligned} z_0 &:= 0 \\ z_i &:= b_i - c_{i-1i} + z_{i-1} - x_i \text{ for every } i \in \{1, \dots, n\}. \end{aligned}$$

It is readily checked that  $z_i = \sum_{k=1}^i b_k - \sum_{k=1}^i c_{k-1k} - \sum_{k=1}^i x_k$  for all  $i \in \{1, \dots, n\}$ . Hence,

$$z_n = b(N) - \sum_{k=1}^n c_{k-1k} - x(N) = b(N) - \sum_{k=1}^n c_{k-1k} - (b(N) - \sum_{k=1}^{n+1} c_{k-1k}) = c_{nn+1} = c_{0n}$$

where the second equality follows from  $x(N) = v_{\hat{\pi}}(N)$ . Consequently, equation (3.5) holds.

With respect to equation (3.6), clearly  $x \geq 0$  implies  $z_i - z_{i-1} \leq b_i - c_{i-1i}$ .

Next, we will show equation (3.7), i.e.,  $z_j - z_i \geq c_{jj+1} - c_{ij+1}$  for all  $i, j \in \{0, 1, \dots, n\}$  with  $i < j$ . Suppose there exist  $i, j \in \{0, 1, \dots, n\}$  with  $i < j$  be such that  $z_j - z_i < c_{jj+1} - c_{ij+1}$ . Define  $S = \{i+1, \dots, j\}$ . Clearly, if we can show that  $x(N \setminus S) < v_{\hat{\pi}}(N \setminus S)$  we will arrive at a contradiction with one of the assumptions in (ii). Indeed,

$$\begin{aligned}
x(N \setminus S) &= b(N \setminus S) - [c_{01} + \dots + c_{i-1i} + c_{jj+1} + \dots + c_{n-1n}] \\
&\quad + [z_0 + \dots + z_{i-1} + z_j + \dots + z_{n-1}] - [z_1 + \dots + z_i + z_{j+1} + \dots + z_n] \\
&= b(N \setminus S) - [c_{01} + \dots + c_{i-1i} + c_{jj+1} + \dots + c_{n-1n}] + z_j - z_i - z_n \\
&= b(N \setminus S) - [c_{01} + \dots + c_{i-1i} + c_{jj+1} + \dots + c_{n-1n} + c_{n0}] + z_j - z_i \\
&< b(N \setminus S) - [c_{01} + \dots + c_{i-1i} + c_{jj+1} + \dots + c_{n-1n} + c_{n0}] + c_{jj+1} - c_{ij+1} \\
&= b(N \setminus S) - [c_{01} + \dots + c_{i-1i} + c_{ij+1} + \dots + c_{n-1n} + c_{n0}] \\
&= b(N \setminus S) - c(\hat{\pi}_{N \setminus S}, N \setminus S) \\
&\leq v_{\hat{\pi}}(N \setminus S),
\end{aligned}$$

where the second equality follows from  $z_0 = 0$ , the third one is a consequence of  $z_n = c_{0n}$ , the strict inequality follows from the assumption and the weak inequality is by definition of  $v_{\hat{\pi}}$ .

(iii) $\Rightarrow$ (i) Let  $z \in \mathbb{R}^{N_0}$  satisfy the conditions (3.5), (3.6), and (3.7) mentioned in assertion (iii) of the theorem. Define  $x_i := b_i - c_{i-1i} + z_{i-1} - z_i$  for all  $1 \leq i \leq n$ . It is readily checked that  $x(N) = v_{\hat{\pi}}(N)$ . Let  $S \subset N$  be a coalition, and let  $R := \{i_1, \dots, i_r\} \subset S$  be such that  $v_{\hat{\pi}}(S) = b(R) - c(\hat{\pi}_R, R)$ . It suffices to prove that  $x(S) \geq v_{\hat{\pi}}(S)$ . For this, note that

$$\begin{aligned}
x(S) &\geq x(R) = b(R) - \sum_{k=1}^r c_{i_k-1i_k} + \sum_{k=1}^r z_{i_k-1} - \sum_{k=1}^r z_{i_k} \\
&= b(R) - \sum_{k=1}^r c_{i_k-1i_k} + \sum_{k=2}^r z_{i_k-1} - \sum_{k=1}^{r-1} z_{i_k} - z_0 + z_{i_1-1} - z_{i_r} + z_n - c_{0n} \\
&\geq b(R) - \sum_{k=1}^r c_{i_k-1i_k} + \sum_{k=2}^r z_{i_k-1} - \sum_{k=1}^{r-1} z_{i_k} + c_{i_1-1i_1} - c_{0i_1} - c_{0i_r} \\
&= b(R) - \sum_{k=1}^r c_{i_k-1i_k} + \sum_{k=1}^{r-1} z_{i_{k+1}-1} - \sum_{k=1}^{r-1} z_{i_k} + c_{i_1-1i_1} - c_{0i_1} - c_{0i_r} \\
&= b(R) - \sum_{k=2}^r c_{i_k-1i_k} + \sum_{k=1}^{r-1} [z_{i_{k+1}-1} - z_{i_k}] - c_{0i_1} - c_{0i_r} \\
&\geq b(R) - \sum_{k=2}^r c_{i_k-1i_k} + \sum_{k=1}^{r-1} [c_{i_{k+1}-1i_{k+1}} - c_{i_k i_{k+1}}] - c_{0i_1} - c_{0i_r}
\end{aligned}$$

$$\begin{aligned}
&= b(R) - \sum_{k=1}^{r-1} c_{i_k i_{k+1}} - c_{0i_1} - c_{0i_r} \\
&= b(R) - c(\hat{\pi}_R, R) \\
&= v_{\hat{\pi}}(S).
\end{aligned}$$

Here, the first inequality is a consequence of equation (3.6) which implies  $x_i \geq 0$  for every  $i \in N$ , the second inequality follows by applying equation (3.7) to  $j = i_1 - 1$ ,  $i = 0$  and to  $j = n$ ,  $i = i_r$  and the second one is also an immediate consequence of equation (3.7).  $\square$

Consider the vector  $x \in \mathbb{R}^N$  defined recursively by

$$x_i = v_{\hat{\pi}}(N) - \max_{k \leq i} \{v_{\hat{\pi}}(N \setminus \{k, \dots, i\}) + x(\{k, \dots, i-1\})\}. \quad (3.8)$$

for  $i \in \{1, \dots, n\}$ . This allocation can be interpreted as follows. Assume only connected coalitions (i.e., coalitions  $\{k, k+1, \dots, i\}$ ) are allowed to step out of the negotiations on the allocation of  $v_{\hat{\pi}}(N)$  and stepping out is decided recursively by the individual players. Consider that player  $i$  wants to step out. If the coalition  $\{k, k+1, \dots, i\}$  decides to step out, the players in  $N \setminus \{k, k+1, \dots, i\}$  will further negotiate the allocation of  $v_{\hat{\pi}}(N \setminus \{k, k+1, \dots, i\})$  and each player  $j \in \{k, k+1, \dots, i-1\}$  already got  $x_j$ . Hence, player  $i$  will be left with  $v_{\hat{\pi}}(N) - [v_{\hat{\pi}}(N \setminus \{k, \dots, i\}) + x(\{k, \dots, i-1\})]$ . Having no influence on “earlier” stepping out player  $i$  can only claim the minimum compensation over the set of all possible connected coalitions  $\{k, \dots, i\}$  with  $1 \leq k \leq i$  which is reflected in (3.8).

It turns out that the allocation  $x$  defined by equation (3.8) is a core element of  $v_{\hat{\pi}}$ . This result is an immediate consequence of the description of the core by coalitions  $N \setminus \{k, \dots, i\}$  given in Theorem 3.3 and Theorem 4 in Derks and Kuipers (1997). Hence, the proof is omitted.

**Theorem 3.4.** Let  $(N, v_{\hat{\pi}})$  be a routing game with revenues corresponding to travel cost matrix  $C \in \mathbb{R}^{N_0 \times N_0}$  and revenue vector  $b \in \mathbb{R}^N$ . Let  $x$  be defined as in equation (3.8). Then,  $x \in \text{Core}(v_{\hat{\pi}})$ .

## 4 Conclusions

In this paper we have analyzed the core of routing games with revenues in which the predetermined route is optimal for the associated combinatorial problem and visits all cities in  $N$ . Next, we provide an example that illustrates that our assumption (2.1) (i.e., the salesman visits all cities) is not restrictive. It turns out that if the salesman only visits some of the cities, those that are unvisited will receive a payoff of zero in any core allocation and the various results provided in the previous sections are still valid.

**Example 4.1.** Consider the routing problem with revenues represented in Figure 5 where the numbers at the edges represent the traveling costs and the boldface numbers at the nodes represent the revenues.

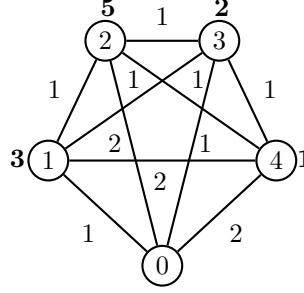


Figure 5: The routing problem with revenues in Example 4.1.

It is readily seen that the optimal tour for this situation only visits the cities 1, 2, and 3 in the order  $0-1-2-3-0$  denoted by  $\hat{\pi}$ . Hence the coalitional values of the routing game are:  $v_{\hat{\pi}}(\{1\}) = 1$ ,  $v_{\hat{\pi}}(\{2\}) = 1$ ,  $v_{\hat{\pi}}(\{3\}) = 0$ ,  $v_{\hat{\pi}}(\{4\}) = 0$ ,  $v_{\hat{\pi}}(\{1, 2\}) = 4$ ,  $v_{\hat{\pi}}(\{1, 3\}) = 2$ ,  $v_{\hat{\pi}}(\{1, 4\}) = 1$ ,  $v_{\hat{\pi}}(\{2, 3\}) = 3$ ,  $v_{\hat{\pi}}(\{2, 4\}) = 1$ ,  $v_{\hat{\pi}}(\{3, 4\}) = 0$ ,  $v_{\hat{\pi}}(\{1, 2, 3\}) = 6$ ,  $v_{\hat{\pi}}(\{1, 2, 4\}) = 4$ ,  $v_{\hat{\pi}}(\{1, 3, 4\}) = 2$ ,  $v_{\hat{\pi}}(\{2, 3, 4\}) = 3$  and  $v_{\hat{\pi}}(N) = 6$ . Here, player 4 is a zero player and the core of the game is  $\text{Core}(v_{\hat{\pi}}) = \text{conv}\{(3, 3, 0, 0), (3, 1, 2, 0), (2, 4, 0, 0), (1, 4, 1, 0), (1, 3, 2, 0)\}$ . Note that the core can still be described by means of the cost of the tour, the vector of revenues, and a vector of compensations as in Theorem 3.3.  $\diamond$

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